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STRUCTURES OF THE DIFFERENCE-TYPE SOLUTIONS OF THE AXISYMMETRIC PROBLEMS FOR ELASTIC BODIES OF FINITE DIMENSIONS*

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Structures of solutions of the first and mixed fundamental problem of the theory of elasticity are constructed for solids of revolution of finite dimensions. The structural formulas are represented by differential operators, which are later replaced by difference operators. Special type contraction formulas are used, constructed with the help of equations, describing the region boundaries, normalized to the first order.

1. Problems of the theory of elasticity can be reduced to integrating the Lamé equations with the corresponding boundary conditions /1/:

a) first fundamental problem

$$\sigma_\nu = f_1^\circ(r, z), \quad \tau_\nu = f_2^\circ(r, z), \quad (r, z) \in \partial\Omega \tag{1.1}$$

b) mixed problem

$$u_r = g_1^\circ(r, z), \quad u_z = g_2^\circ(r, z), \quad (r, z) \in \partial\Omega_1 \tag{1.2}$$

$$\sigma_\nu = f_1^\circ(r, z), \ \tau_\nu = f_2^\circ(r, z), \ (r, z) \in \partial\Omega_2 \tag{1.3}$$

Here u_r and u_z are components of the displacement vector, σ_ν, τ_ν denote the normal and shear stress, $f_1^\circ, f_2^\circ, g_1^\circ, g_2^\circ$ are given functions and $\partial\Omega$ is the boundary of the elastic body Ω ($\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$). The left-hand parts of the relations (1.1), (1.3) can be written in terms of the components u_r and u_z as follows:

$$\sigma_\nu = (\lambda + 2\mu)\left(\frac{\partial u_r}{\partial\nu} l + \frac{\partial u_z}{\partial\nu} m\right) + \lambda\left(\frac{\partial u_r}{\partial\tau} m - \frac{\partial u_z}{\partial\tau} l + \frac{u_r}{r}\right) \tag{1.4}$$

$$\tau_\nu = \mu\left(\frac{\partial u_r}{\partial\nu} m - \frac{\partial u_z}{\partial\nu} l + \frac{\partial u_r}{\partial\tau} l + \frac{\partial u_z}{\partial\tau} m\right)$$

where ν and τ is the outward normal and tangent to $\partial\Omega$, and l, m are the direction cosines of the normal. We assume that the principal vector and principal moment of the elastic systems under consideration are both zero.

Let us construct the structures of solutions to the above problems, i.e. let us find the expressions for the displacement vector components in such a form, that the corresponding boundary conditions are satisfied exactly. Following /2/, we denote by $\omega(r, z) = 0$ the equation for $\partial\Omega$ normalized to the first order, i.e.

$$\omega|_{\partial\Omega} = 0, \ |\nabla\omega|_{\partial\Omega} = \partial\omega/\partial\nu|_{\partial\Omega} = 1 \tag{1.5}$$

The operators D_1 and T_1 introduced in /2/

$$D_1 = \frac{\partial\omega}{\partial r}\frac{\partial}{\partial r} + \frac{\partial\omega}{\partial z}\frac{\partial}{\partial z}, \quad T_1 = -\frac{\partial\omega}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\omega}{\partial r}\frac{\partial}{\partial z} \tag{1.6}$$

are defined in the region $\Omega \cup \partial\Omega$, and the following formulas hold for them on $\partial\Omega$:

$$D_1 f = -\frac{\partial f}{\partial\nu}, \quad T_1 f = \frac{\partial f}{\partial\tau}, \quad D_1(\omega f) = f, \quad T_1(\omega f) = 0 \tag{1.7}$$

$$D_1 F(t) = \sum_{i=1}^{2} D_1 t_i \frac{\partial}{\partial t_i} F(t), \quad f \in C^1(\Omega), \quad F \in C^1(\Omega)$$

$$(t(r, z) = [t_1(r, z), t_2(r, z)])$$

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In what follows, we shall utilize the following formulas /3/:

$$\begin{aligned}
&(Q_h - 1) f(r, z) = -\omega D_1 f(r, z) + O(\omega^2) \\
&(Q_\tau - 1) f(r, z) = -\omega T_1 f(r, z) + O(\omega^2) \\
&h_r = \omega(r_-, z) - \omega(r_+, z),\ h_z = \omega(r, z_-) - \omega(r, z_+) \\
&h = (h_r, h_z),\quad \tau = (-h_z, h_r) \\
&Q_h f(r, z) = f(r + h_r, z + h_z),\ Q_\tau f((r, z) = f(r - h_z, z + h_r) \\
&r_\pm = r \pm {}^1/_2 \omega(r, z),\quad z_\pm = z \pm {}^1/_2\, \omega(r, z)
\end{aligned} \tag{1.8}$$

Let us introduce the contraction formulas

$$\begin{aligned}
&\theta_{h_p}{}^{(s, q)} = \omega(L_{-h_p}^{(s, q)}) - \omega(L_{+h_p}^{(s, q)}) \\
&s = r, z;\ q = r, z, 0;\ p = r, z \\
&L_{\pm h_p}^{(s, r)} = (r \pm {}^1/_2 \theta_{h_p}^{(s, 0)}, z),\ L_{\pm h_p}^{(s, z)} = (r, z \pm {}^1/_2\, \theta_{h_p}^{(s, 0)}) \\
&L_{\pm h_p}^{(r, 0)} = (r \pm {}^1/_2 h_p, z),\ L_{\pm h_p}^{(z, 0)} = (r, z \pm {}^1/_2 h_p)
\end{aligned} \tag{1.9}$$

It can be shown for (1.9) that

$$D_1 \theta_{h_p}^{(s, q)} = -\frac{\partial\omega}{\partial s}\frac{\partial\omega}{\partial q}\frac{\partial\omega}{\partial p} \tag{1.10}$$

Using the properties (1.7) of the operator D_1, we can write

$$D_1\theta_{h_p}^{(s, q)} = D_1[\omega(L_{-h_p}^{(s, q)}) - \omega(L_{+h_p}^{(s, q)})] = D_1\omega(L_{-h_p}^{(s, q)}) - D_1\omega(L_{+h_p}^{(s, q)}) = \left[\frac{\partial\omega(L_{-h_p}^{(s, q)})}{\partial r} - \frac{\partial\omega(L_{+h_p}^{(s, q)})}{\partial r}\right]\frac{\partial\omega}{\partial r} +$$

$$\left[\frac{\partial\omega(L_{-h_p}^{(s, q)})}{\partial z} - \frac{\partial\omega(L_{+h_p}^{(s, q)})}{\partial z}\right]\frac{\partial\omega}{\partial z} - \frac{1}{2} D_1\theta_{h_p}^{(s, q)}\left[\frac{\partial\omega(L_{h_p}^{(s, q)})}{\partial q} + \frac{\partial\omega(L_{+h_p}^{(s, q)})}{\partial q}\right]$$

and similarly

$$D_1\theta_{h_p}^{(s, 0)} = \left[\frac{\partial\omega(L_{-h_p}^{(s, 0)})}{\partial r} - \frac{\partial\omega(L_{+h_p}^{(s, 0)})}{\partial r}\right]\frac{\partial\omega}{\partial r} +$$

$$\left[\frac{\partial\omega(L_{-h_p}^{(s, 0)})}{\partial z} - \frac{\partial\omega(L_{+h_p}^{(s, 0)})}{\partial z}\right]\frac{\partial\omega}{\partial z} - \frac{1}{2} D_1 h_p\left[\frac{\partial\omega(L_{-h_p}^{(s, 0)})}{\partial s} + \frac{\partial\omega(L_{+h_p}^{(s, 0)})}{\partial s}\right],\quad D_1 h_p = -\frac{\partial\omega}{\partial p} D_1\omega$$

Passing to the limit in the expressions obtained as $\omega(r, z) \to 0$ and taking into account the fact that $\omega(r, z)$ has continuous derivatives, we obtain

$$D_1\theta_{h_p}^{(s, q)} = -D_1\theta_{h_p}^{(s, 0)}\frac{\partial\omega}{\partial q},\quad D_1\theta_{h_p}^{(s, 0)} = -D_1 h_p\frac{\partial\omega}{\partial s},\quad D_1 h_p = -\frac{\partial\omega}{\partial p} \tag{1.11}$$

from which we can obtain (1.10) by direct substitution. Using the formula (1.10), we obtain

$$\theta_{h_p}^{(s, q)} = -\omega\frac{\partial\omega}{\partial s}\frac{\partial\omega}{\partial q}\frac{\partial\omega}{\partial p} + O(\omega^2)$$

and following the methods of /3/ we write the difference operators

$$\begin{aligned}
&(Q_h^{(s, q)} - 1) f(r, z) = -\omega\frac{\partial\omega}{\partial s}\frac{\partial\omega}{\partial q} D_1 f(r, z) + O(\omega^2) \\
&(Q_\tau^{(s, q)} - 1) f(r, z) = -\omega\frac{\partial\omega}{\partial s}\frac{\partial\omega}{\partial q} T_1 f(r, z) + O(\omega^2) \\
&Q_h{}^{(s, q)} f(r, z) = f(r + \theta_{h_z}{}^{(s, q)}, z + \theta_{h_z}{}^{(s, q)}) \\
&Q_\tau{}^{(s, q)} f(r, z) = f(r - \theta_{h_z}{}^{(s, q)}, z + \theta_{h_r}{}^{(s, q)})
\end{aligned} \tag{1.12}$$

2. We find here the structure of the solution of the first fundamental problem. Using the operators (1.6), we extend the boundary conditions (1.1) to the region Ω

$$\begin{aligned}
&(\lambda + 2\mu)\left(\frac{\partial\omega}{\partial r} D_1 u_r + \frac{\partial\omega}{\partial z} D_1 u_z\right) + \lambda\left(\frac{\partial\omega}{\partial r} T_1 u_z - \frac{\partial\omega}{\partial z} T_1 u_r + \frac{u_r}{r}\right) = f_1 + \omega\varphi_{11} \\
&\mu\left(\frac{\partial\omega}{\partial z} D_1 u_r - \frac{\partial\omega}{\partial r} D_1 u_z - \frac{\partial\omega}{\partial r} T_1 u_r - \frac{\partial\omega}{\partial z} T_1 u_z\right) = f_2 + \omega\varphi_{21}
\end{aligned} \tag{2.1}$$

Here f_1 and f_2 denote the continuations of the functions $f_1{}^\circ$ and $f_2{}^\circ$ into the region Ω. The continuation can be carried out with help of the formulas given in /2/, and $\varphi_{11}, \varphi_{21}$ are undefined functions. We shall seek the structure of the solution in the form

$$u_r = \Phi_{11} + \omega\Phi_{12}^{\circ}, \quad u_z = \Phi_{21} + \omega\Phi_{22}^{\circ} \tag{2.2}$$

Substituting (2.2) into (2.1) and carrying out certain transformations based on the linearity properties of the operators D_1 and T_1, we obtain the following system in terms of $\Phi_{12}^{\circ}, \Phi_{22}^{\circ}$:

$$(\lambda + 2\mu)\left(\Phi_{12}^{\circ}\frac{\partial\omega}{\partial r} + \Phi_{22}^{\circ}\frac{\partial\omega}{\partial z}\right) = \Psi_1 + \omega\varphi_{12} \tag{2.3}$$

$$\mu\left(\Phi_{12}^{\circ}\frac{\partial\omega}{\partial z} - \Phi_{22}^{\circ}\frac{\partial\omega}{\partial r}\right) = \Psi_2 + \omega\varphi_{22}$$

$$\Psi_1 = f_1 - (\lambda + 2\mu)\left(\frac{\partial\omega}{\partial r}D_1\Phi_{11} + \frac{\partial\omega}{\partial z}D_1\Phi_{21}\right) - \lambda\left(\frac{\partial\omega}{\partial r}T_1\Phi_{21} - \frac{\partial\omega}{\partial z}T_1\Phi_{11} + \frac{\Phi_{11}}{r}\right)$$

$$\Psi_2 = f_2 - \mu\left(\frac{\partial\omega}{\partial z}D_1\Phi_{11} - \frac{\partial\omega}{\partial r}D_1\Phi_{21} - \frac{\partial\omega}{\partial r}T_1\Phi_{11} - \frac{\partial\omega}{\partial z}T_1\Phi_{21}\right)$$

where φ_{12} and φ_{22} are new undefined functions. From (2.3) we obtain the expressions for Φ_{12}°, Φ_{22}°, and this enables us to write the structure of the solution (2.2) in the form

$$\begin{aligned}u_r = {} & \frac{1}{\lambda+2\mu}\omega\frac{\partial\omega}{\partial r}f_1 + \frac{1}{\mu}\omega\frac{\partial\omega}{\partial z}f_2 + \Phi_{11} - \omega D_1\Phi_{11} + \\ & \omega\frac{\partial\omega}{\partial z}\frac{\partial\omega}{\partial r}T_1\Phi_{11} + \frac{\lambda}{\lambda+2\mu}\omega\frac{\partial\omega}{\partial r}\frac{\partial\omega}{\partial z}T_1\Phi_{11} - \\ & \frac{\lambda}{\lambda+2\mu}\omega\left(\frac{\partial\omega}{\partial r}\right)^2 T_1\Phi_{21} + \omega\left(\frac{\partial\omega}{\partial z}\right)^2 T_1\Phi_{21} - \frac{\lambda}{\lambda+2\mu}\omega\frac{\partial\omega}{\partial r}\frac{\Phi_{11}}{r} + \omega^2\Phi_{12}\end{aligned} \tag{2.4}$$

$$\begin{aligned}u_z = {} & -\frac{1}{\mu}\omega\frac{\partial\omega}{\partial r}f_2 + \frac{1}{\lambda+2\mu}\omega\frac{\partial\omega}{\partial z}f_1 + \Phi_{21} - \omega D_1\Phi_{21} - \\ & \omega\frac{\partial\omega}{\partial r}\frac{\partial\omega}{\partial z}T_1\Phi_{21} - \frac{\lambda}{\lambda+2\mu}\omega\frac{\partial\omega}{\partial z}\frac{\partial\omega}{\partial r}T_1\Phi_{21} - \omega\left(\frac{\partial\omega}{\partial r}\right)^2 T_1\Phi_{11} + \\ & \frac{\lambda}{\lambda+2\mu}\omega\left(\frac{\partial\omega}{\partial z}\right)^2 T_1\Phi_{11} - \frac{\lambda}{\lambda+2\mu}\omega\frac{\partial\omega}{\partial z}\frac{\Phi_{11}}{r} + \omega^2\Phi_{22}\end{aligned}$$

where $\Phi_{ij}\ (i, j = 1, 2)$ are the undefined components of the structure. Replacing in (2.4) the expressions

$$\omega\frac{\partial\omega}{\partial p},\ \omega D_1\Phi_{i1},\ \omega\frac{\partial\omega}{\partial s}\frac{\partial\omega}{\partial q}T_1\Phi_{i1}\ (i = 1, 2;\ s = r, z;\ q = r, z;\ p = r, z)$$

by the difference operators (1.8), (1.12), we can write

$$\begin{aligned}u_r = {} & -\frac{1}{\lambda+2\mu}h_r f_1 - \frac{1}{\mu}h_z f_2 + \Phi_{11} + (Q_h - 1)\Phi_{11} - \\ & (Q_\tau^{(z,r)} - 1)\Phi_{11} - \frac{\lambda}{\lambda+2\mu}(Q_\tau^{(r,z)} - 1)\Phi_{11} + \\ & \frac{\lambda}{\lambda+2\mu}(Q_\tau^{(r,r)} - 1)\Phi_{21} - (Q_\tau^{(z,z)} - 1)\Phi_{21} + \frac{\lambda}{\lambda+2\mu}h_r\frac{\Phi_{11}}{r} + \omega^2\Phi_{12}\end{aligned}$$

$$\begin{aligned}u_z = {} & \frac{1}{\mu}h_r f_2 - \frac{1}{\lambda+2\mu}h_z f_1 + \Phi_{21} + (Q_h - 1)\Phi_{21} + \\ & (Q_\tau^{(r,z)} - 1)\Phi_{21} + \frac{\lambda}{\lambda+2\mu}(Q_\tau^{(z,r)} - 1)\Phi_{21} + (Q_\tau^{(r,r)} - 1)\Phi_{11} - \\ & \frac{\lambda}{\lambda+2\mu}(Q_\tau^{(z,z)} - 1)\Phi_{11} + \frac{\lambda}{\lambda+2\mu}h_z\frac{\Phi_{11}}{r} + \omega^2\Phi_{22}\end{aligned}$$

3. For the mixed problem the structures of the solutions can be written in the form

$$u_r = g_1 + \omega_1\Phi_{11} + \omega\Phi_{12}^{\circ}, \quad u_z = g_2 + \omega_1\Phi_{21} + \omega\Phi_{22}^{\circ} \tag{3.1}$$

where g_1 and g_2 are the continuations of the functions g_1° and g_2° into the region Ω, $\omega_1 = 0$ is the equation of the segment $\partial\Omega_1$, $\Phi_{11}, \Phi_{21}, \Phi_{12}^{\circ}, \Phi_{22}^{\circ}$ are undefined functions. The structure (3.1) takes into account the boundary conditions (1.2).

Denoting by $\omega_2 = 0$ the equation of the segment $\partial\Omega_2$ normalized to the first order, we extend the boundary conditions (1.3) into $\Omega \cup \partial\Omega_1$:

$$(\lambda + 2\mu)\left(\frac{\partial\omega_2}{\partial r}D_1^{(2)}u_r + \frac{\partial\omega_2}{\partial z}D_1^{(2)}u_z\right) + \lambda\left(\frac{\partial\omega_2}{\partial r}T_1^{(2)}u_z - \frac{\partial\omega_2}{\partial z}T_1^{(2)}u_r + \frac{u_r}{r}\right) = f_1 + \omega_2\varphi_{11} \tag{3.2}$$

$$\mu\left(\frac{\partial\omega_2}{\partial z}D_1^{(2)}u_r - \frac{\partial\omega_2}{\partial r}D_1^{(2)}u_z - \frac{\partial\omega_2}{\partial r}T_1^{(2)}u_r - \frac{\partial\omega_2}{\partial z}T_1^{(2)}u_z\right) = f_2 + \omega_2\varphi_{21}$$

Here φ_{11} and φ_{21} are arbitrary functions. In (3.2) and in what follows, the index 2 within the round brackets means that the operators (differential and difference) are taken with respect to the function ω_2.

Substituting the expressions for u_r and u_z from (3.1) into (3.2) and repeating the manipulations carried out when constructing the structure of solutions for the first fundamental problem, we obtain

$$\begin{aligned}
\Phi_{12}{}^\circ &= \frac{1}{\lambda+2\mu}\frac{\partial\omega_2}{\partial r} f_1 + \frac{1}{\mu}\frac{\partial\omega_2}{\partial z} f_2 - D_1^{(2)}\Phi_{11}{}^\circ + \frac{\partial\omega_2}{\partial z}\frac{\partial\omega_2}{\partial r} T_1^{(2)}\Phi_{11}{}^\circ + \\
&\frac{\lambda}{\lambda+2\mu}\frac{\partial\omega_2}{\partial r}\frac{\partial\omega_2}{\partial z} T_1^{(2)}\Phi_{11}{}^\circ - \frac{\lambda}{\lambda+2\mu}\left(\frac{\partial\omega_2}{\partial r}\right)^2 T_1^{(2)}\Phi_{21}{}^\circ + \\
&\left(\frac{\partial\omega_2}{\partial z}\right)^2 T_1^{(2)}\Phi_{21}{}^\circ - \frac{\lambda}{\lambda+2\mu}\frac{\partial\omega_2}{\partial r}\frac{\Phi_{11}{}^\circ}{r} + \omega_2\Phi_{12} \\
\Phi_{22}{}^\circ &= -\frac{1}{\mu}\frac{\partial\omega_2}{\partial r} f_2 + \frac{1}{\lambda+2\mu}\frac{\partial\omega_2}{\partial z} f_1 - D_1^{(2)}\Phi_{21}{}^\circ - \frac{\partial\omega_2}{\partial r}\frac{\partial\omega_2}{\partial z} \times \\
&T_1^{(2)}\Phi_{21}{}^\circ - \frac{\lambda}{\lambda+2\mu}\frac{\partial\omega_2}{\partial z}\frac{\partial\omega_2}{\partial r} T_1^{(2)}\Phi_{21}{}^\circ - \left(\frac{\partial\omega_2}{\partial r}\right)^2 T_1^{(2)}\Phi_{11}{}^\circ + \\
&\frac{\lambda}{\lambda+2\mu}\left(\frac{\partial\omega_2}{\partial z}\right)^2 T_1^{(2)}\Phi_{11}{}^\circ - \frac{\lambda}{\lambda+2\mu}\frac{\partial\omega_2}{\partial z}\frac{\Phi_{11}{}^\circ}{r} + \omega_2\Phi_{22} \\
\Phi_{11}{}^\circ &= g_1 + \omega_1\Phi_{11}, \quad \Phi_{21}{}^\circ = g_2 + \omega_1\Phi_{21}
\end{aligned} \tag{3.3}$$

where Φ_{12}, Φ_{22} are the undefined components of the structure.

According to (3.1) and (3.3), the solution structure of the mixed problem written in terms of the difference operators (1.8), (1.12), will have the form

$$\begin{aligned}
u_r &= g_1 - \frac{1}{\lambda+2\mu} h_r^{(2)} f_1 - \frac{1}{\mu} h_z^{(2)} f_2 + \omega_1\Phi_{11} + (Q_{h(2)} - 1)\Phi_{11}{}^\circ - \\
&(Q_{\tau(2)}^{(z,r)} - 1)\Phi_{11}{}^\circ - \frac{\lambda}{\lambda+2\mu}(Q_{\tau(2)}^{(r,z)} - 1)\Phi_{11}{}^\circ + \frac{\lambda}{\lambda+2\mu}(Q_{\tau(2)}^{(r,r)} - 1)\Phi_{21}{}^\circ - \\
&(Q_{\tau(2)}^{(z,z)} - 1)\Phi_{21}{}^\circ + \frac{\lambda}{\lambda+2\mu} h_r^{(2)} \frac{\Phi_{11}{}^\circ}{r} + \omega\omega_2\Phi_{12} \\
u_z &= g_2 + \frac{1}{\mu} h_r^{(2)} f_2 - \frac{1}{\lambda+2\mu} h_z^{(2)} f_1 + \omega_1\Phi_{21} + (Q_{h(2)} - 1)\Phi_{21}{}^\circ + \\
&(Q_{\tau(2)}^{(r,z)} - 1)\Phi_{21}{}^\circ + \frac{\lambda}{\lambda+2\mu}(Q_{\tau(2)}^{(z,r)} - 1)\Phi_{21}{}^\circ + (Q_{\tau(2)}^{(r,r)} - 1)\Phi_{11}{}^\circ - \\
&\frac{\lambda}{\lambda+2\mu}(Q_{\tau(2)}^{(z,z)} - 1)\Phi_{11}{}^\circ + \frac{\lambda}{\lambda+2\mu} h_z^{(2)} \frac{\Phi_{11}{}^\circ}{r} + \omega\omega_2\Phi_{22}
\end{aligned}$$

The structural formulas constructed contain a number of arbitrary functions which can be chosen so as to satisfy the Lamé equations inside the region Ω and to allow due regard to the specific features of the solutions (action of concentrated forces /1/, influence of the incoming angles and lines of change in the boundary conditions /4/, etc.). The difference-type structures corresponding to the boundary conditions of axisymmetric problems of the theory of elasticity were realized in the program generator "POLE-3" of the Institute of Mechanical Engineering Problems, Akad. Nauk USSR.

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